TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 9, September 1996

COMBINATORIAL B_n-ANALOGUES OF SCHUBERT POLYNOMIALS

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Abstract. Combinatorial B_n-analogues of Schubert polynomials and corresponding symmetric functions are constructed and studied. The development is based on an exponential solution of the type B Yang-Baxter equation that involves the nilCoxeter algebra of the hyperoctahedral group.

0. Introduction

This paper is devoted to the problem of constructing type B analogues of the Schubert polynomials of Lascoux and Schützenberger (see, e.g., [L2], [M] and the literature therein). We begin with reviewing the basic properties of the type A polynomials, stating them in a form that would allow subsequent generalizations to other types.

Let $W = W_n$ be the Coxeter group of type A_n with generators $s_1, \ldots, s_n$ (that is, the symmetric group S_{n+1}). Let $x_1, x_2, \ldots$ be formal variables. Then W naturally acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$ by permuting the variables. Let I_W denote the ideal generated by homogeneous non-constant W-symmetric polynomials. By a classical result [Bo], the cohomology ring $H(F)$ of the flag variety of type A can be canonically identified with the quotient $\mathbb{C}[x_1, \ldots, x_{n+1}]/I_W$. This ring is graded by the degree and has a distinguished linear basis of homogeneous cosets X_w modulo I_W, labelled by the elements w of the group. Let us state for the record that, for an element $w \in W$ of length $l(w)$,

(0) $\quad X_w$ is a homogeneous polynomial of degree $l(w)$; $\quad X_1 = 1$;

the latter condition signifies proper normalization.

As shown by Bernstein, Gelfand, and Gelfand [BGG], one can construct such a basis using the *divided difference* operators ∂_i acting in the polynomial ring. Namely, define an operator ∂_i associated with a generator s_i by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}} \quad , \quad i = 1, 2, \ldots .$$

Then recursively define, for each element $w \in W$, a polynomial X_w by

$$\partial_i X_{ws_i} = X_w \quad \text{if} \quad l(ws_i) = l(w) + 1 . \tag{1}$$

The recursion starts at the "top polynomial" X_{w_0} that corresponds to the element $w_0 \in W$ of maximal length. Choose X_{w_0} to be an arbitrary and sufficiently generic

Received by the editors January 6, 1994.
1991 *Mathematics Subject Classification.* Primary 05E15; Secondary 05E05, 14M15.
Key words and phrases. Yang-Baxter equation, Schubert polynomials, symmetric functions.
Partially supported by the NSF (DMS-9400914).

homogeneous polynomial of degree $l(w_0)$. Then the X_w are well defined by (1), and provide a basis for the quotient ring $H(F)$. The normalization condition $X_1 = 1$ (see (0)) can be ensured by multiplying all the X_w by an appropriate constant. The basis of the quotient ring thus obtained is canonical in the sense that it does not depend (modulo the ideal I_W) on the particular choice of X_{w_0}. One can as well replace the recursion (1) by a weaker condition

$$\partial_i X_{ws_i} \equiv X_w \bmod I_w \quad \text{if} \quad l(ws_i) = l(w) + 1 \tag{1a}$$

which is exactly equivalent to saying that the X_w represent the right cohomology classes.

In order to be able to calculate in the cohomology ring, one would also like to know how the basis elements multiply, i.e., find the structure constants c_{uv}^w such that $X_u X_v = \sum_w c_{uv}^w X_w \bmod I_W$. Lascoux and Schützenberger [LS] discovered that a particular choice of the top element (viz., $X_{w_0} = x_1^{n-1} \cdots x_{n-2}^2 x_{n-1}$) makes it possible to get rid of the unpleasant "mod I_w" provision from the last identity. More precisely, they constructed a family of polynomials X_w (called the type A Schubert polynomials and denoted $\mathfrak{S}_w$ elsewhere in this paper) which satisfy (0)-(1) and have the following property:

(2) for any $u, v \in W_n$ and for a sufficiently large m,

$$X_u X_v = \sum_{w \in W_m} c_{uv}^w X_w \ .$$

Another remarkable property of the Schubert polynomials of Lascoux and Schützenberger that also makes good geometric sense is the following:

(3) the X_w are polynomials with nonnegative integer coefficients.

One can give a direct combinatorial explanation of this phenomenon by providing an alternative definition of the Schubert polynomials in terms of reduced decompositions and compatible sequences [BJS] or, equivalently, via noncommutative generating function in the nilCoxeter algebra of W (see [FS]). This alternative description of the Schubert polynomials that avoids the recurrence process proved to be a helpful tool in deriving their fundamental properties and dealing with their generalizations, such as the Grothendieck polynomials of Lascoux and Schützenberger [L1], [FK2].

It is transparent from the combinatorial formula — and not hard to deduce from the original definition — that the Schubert polynomials are *stable* with respect to a natural embedding $W_n \hookrightarrow W_m$, $n < m$ (as a parabolic subgroup with generators s_0, $\ldots$, s_{n-1}) and the corresponding projection $\mathrm{pr} : \mathbb{C}[x_1, \ldots, x_{m+1}] \to \mathbb{C}[x_1, \ldots, x_{n+1}]$ defined by

$$\mathrm{pr} : f(x_1, \ldots, x_{m+1}) \mapsto f(x_1, \ldots, x_{n+1}, 0, \ldots, 0) \ .$$

In other words,

(4) for $w \in W_n \subset W_m$, $\quad X_w^{(n)} = \mathrm{pr}\, X_w^{(m)}$

where $X_w^{(n)}$ denotes the Schubert polynomial for w treated as an element of W_n, and pr simply takes the last $m - n$ variables x_i to zero. (Actually, the Schubert polynomials of type A are stable in an even stronger sense, but for the other types we will only require condition (4), as stated above.)

It seems reasonable to attempt to reproduce all of the above for other classical types, and as a first step for type B where $W_n = B_n$ is the hyperoctahedral group. There is a natural action of B_n on the polynomial ring (see Section 1); as before, the cohomology ring can be identified with the quotient $\mathbb{C}[x_1, x_2, \dots]/I_W$, and its basis can be constructed via divided differences in the similar way. (There are some peculiarities related to the definition of the divided differences for type B; see Section 1.)

Ideally, B_n-Schubert polynomials would be a certain family of polynomials that satisfy the verbatim B-analogues of the conditions (0)-(4) above. Unfortunately, such a family of polynomials simply does not exist. We will show in Section 10 that, already for the hyperoctahedral group B_2 with two generators, one cannot find 8 polynomials satisfying all relevant instances of conditions (0)-(3), even if we replace (1) by a weaker condition (1a).

Since having all of (0)-(3) is impossible, we have to sacrifice one of the basic properties. Abandoning condition (0) seems extremely unreasonable. We are then led to the problems of finding polynomials satisfying (0) and two of the properties (1)-(3): (1) and (2), (2) and (3), or (1) and (3). To sweeten the pill, let us also require that (4) be satisfied. We arrived at the following three problems whose solutions could be viewed as B_n-analogues (in the three different senses specified below) of the Schubert polynomials.

Problem 0-1-2-4. (B_n-Schubert polynomials of the second kind.) *Find a family of polynomials $X_w = X^{(n)} = \mathfrak{B}_w^{(n)}(x_1, \dots, x_n)$, one for each element w of each group $W_n = B_n$, which satisfy the type B versions of conditions* (0), (1), (2), *and* (4).

In this problem, conditions (0)-(1) ensure that the X_w represent the corresponding cosets of the B-G-G basis, and condition (2) means that they multiply exactly as the cohomology classes do. In Section 7, we construct a solution to Problem 0-1-2-4 by giving a simple explicit formula for the generating function of the X_w in the nilCoxeter algebra of the hyperoctahedral group. We also prove that the stable limits of our polynomials coincide with the power series introduced by Billey and Haiman [BH] whose definition involved a λ-ring substitution and Schur P-functions. This also allows us to replace a long and technical verification of (1) in [BH] by a few lines of a transparent computation.

Problem 0-2-3-4. (B_n-Schubert polynomials of the first kind.) *Find a family of polynomials $X_w = X^{(n)} = \mathfrak{b}_w^{(n)}(x_1, \dots, x_n)$ satisfying the type B versions of conditions* (0), (2), (3), *and* (4).

This problem may at first sight look unnatural since these polynomials no longer represent Schubert cycles. However, if one really wants to compute in the quotient ring $H(F)$, property (1) is not critical, once it is known that the structure constants are correct. On the other hand, the solution of Problem 0-2-3-4 that we suggest in Section 6 is very natural combinatorially and much easier to work with than in the previous case of the polynomials of the second kind. The formulas become much simpler; for example, the Schubert polynomial of the first kind for the element $w_0 \in B_3$ is given by

$$\mathfrak{b}_{w_0}^{(3)} = x_1 x_2^2 x_3^3 (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) \tag{5}$$

whereas, for the same element w_0, the Schubert polynomial of the second kind is

(6)
$$\begin{aligned}\mathfrak{B}^{(3)}_{w_0} =&\frac{1}{512}\Big(-4x_1^6x_2x_3^2 - 12x_1^5x_2^3x_3 + 4x_1^5x_2x_3^3 + 4x_1^4x_2^3x_3^2 - 4x_1^4x_2^2x_3^3 + 4x_1^7x_2x_3\\
&- 2x_1^4x_2^4x_3 - 2x_1^4x_2x_3^4 + 4x_1^3x_2^2x_3^4 + 16x_1^2x_2^4x_3^3 + 4x_1^2x_2^3x_3^4 + 4x_1^2x_2^5x_3^2\\
&- 8x_1^2x_2^2x_3^5 + 8x_1^3x_2^3x_3^3 + 12x_1^3x_2^5x_3 - 4x_1x_2^7x_3 - 12x_1x_2^5x_3^3 - 8x_1^2x_2^6x_3\\
&- 8x_1x_3^6x_2^2 - 4x_1^6x_2^3 + 4x_1^7x_2^2 + 5x_1^8x_2 - 2x_1^5x_2^4 - 8x_1^6x_3^3 + 3x_1^8x_3 - 6x_1^5x_3^4\\
&- 2x_1^4x_2^5 - 4x_1^2x_2^7 - 12x_1^3x_2^6 + 9x_1x_2^8 + 8x_1^3x_3^6 + 6x_1^4x_3^5 - 3x_1x_3^8 - 4x_2^3x_3^6\\
&+ 12x_2^2x_3^7 - 14x_2^4x_3^5 + 5x_2x_3^8 - 4x_2^7x_3^2 - 4x_2^6x_3^3 + 7x_2^8x_3 + 12x_1x_3^7x_2\\
&- 2x_2^5x_3^4 - 20x_1^3x_3^5x_2 + 4x_1x_3^5x_2^2 - 4x_1^2x_3^6x_2 + 2x_2^4x_3^4x_1 + x_1^9 + 5x_2^9 - x_3^9\Big).\end{aligned}$$

We also show (see Section 7) that, in fact, the B_n-Schubert polynomials of the first and second kinds are related to each other by a certain "change of variables." (This explains why the structure constants are the same in both cases.) Thus one can switch between polynomials of the two kinds, if necessary.

Problem 0-1-3-4. (<u>Schubert polynomials of the third kind</u>.) *Construct a family of polynomials X_w satisfying conditions* (0), (1), (3), *and* (4).

This is simply a question of finding explicit "combinatorial" representatives for the cohomology classes. In Section 9, we conjecture[1] a solution of Problem 0-1-3-4 for the type C; the corresponding type B polynomials differ from these by a factor of the form 2^{-k}. Recently, we discovered that our "Schubert polynomials of the third kind" can also be obtained from the polynomials in two sets of variables introduced by Fulton in [Fu], by setting $y_1 = y_2 = \cdots = 0$.

We now briefly describe the general framework of our constructions and the organization of the paper. In [FS], [FK1]-[FK3] an approach to the theory of Schubert polynomials was developed that was based on an exponential solution of the Yang-Baxter equation (YBE) in the nilCoxeter algebra of the symmetric group, the latter being the abstract algebra isomorphic to the algebra of divided differences. In this paper, we adapt this approach to the case of the hyperoctahedral group.

Section 2 presents a straightforward B_n-analogue of the main geometric construction used in [FK1] and inspired by Cherednik's work [Ch]; the role of the YBE is briefly explained. (At this point, an acquaintance with our "A_n-paper" [FK1] would be very helpful.) In Section 3, some exponential solutions of the B_n-YBE are given; we refer to [FK3] for details. Section 4 introduces B_n-symmetric functions (generalized Stanley symmetric functions of type B) which can be associated with any such solution. Type B Schubert expressions of the first kind are introduced in Section 5. For the nilCoxeter algebra solution of the YBE, these expressions give rise to the B_n-Schubert polynomials $\mathfrak{b}^{(n)}_w$ of the first kind which are studied in Section 6. In Section 7, we define the Schubert polynomials $\mathfrak{B}^{(n)}_w$ of the second kind and relate them to the Billey-Haiman construction. In Section 8, we introduce and study the Stanley symmetric functions of type B; this study was continued in

[1] *Note added in proof*. This conjecture is now a theorem. Tao Kai Lam has proved, in November 1995, that our Schubert polynomials of the third kind do indeed satisfy property (3). Properties (0), (1), and (4) are checked in Section 9.

[TKL1], [TKL2] and [BH]. In Section 9, the Schubert polynomials of the third kind are discussed. Section 10 contains the proof that it is impossible to simultaneously satisfy conditions (0)-(3).

We provide tables of the Schubert polynomials of the three kinds for the types B_2 and B_3, except for the table of the B_3-polynomials of the second kind, which would occupy several pages. An 18-page table of the 384 B_4-Schubert polynomials of the first kind was produced by Sébastien Veigneau using his wonderful Maple package ACE; this table is available from the authors upon request.

As earlier in [FK1], we intentionally use in this paper the geometric approach that allows us to derive algebraic identities in the nilCoxeter algebra by modifying, according to certain rules, corresponding configurations of labelled pseudo-lines. A typical example is Theorem 4.4 that is proved by Figure 7. A formal algebraic version of this proof would be a straightforward (albeit messy and unreadable) translation from the geometric language.

The combinatorial constructions of this paper can be adapted to describe the Schubert polynomials of types C and D, reproducing, in particular, the corresponding results in [BH]. A more or less straightforward modification of these constructions leads to combinatorial formulas for the Grothendieck polynomials of types BCD, in the spirit of [FK2], and also to the double Schubert polynomials of respective types. We plan to discuss these generalizations in a separate publication.

Acknowledgements

The authors are grateful to Tao Kai Lam, Peter Magyar, Victor Reiner, Richard Stanley, John Stembridge, and Andrei Zelevinsky for helpful comments.

This paper originally appeared in June 1993 as a preprint PAR-LPTHE 93/33 of Université Paris VI. That earlier version did not contain the construction of the polynomials of the second and third kind; neither could it refer to the later work of Billey and Haiman [BH]. We thank the referee whose legitimate request compelled us to develop the corresponding theory presented in Sections 7 and 9.

1. The hyperoctahedral group: definitions and conventions

The hyperoctahedral group B_n is the group of symmetries of the n-dimensional cube. We define B_n formally as the group with generators $s_0, \ldots, s_{n-1}$ satisfying the relations

$$\begin{aligned} &s_i s_j = s_j s_i \ , \quad |i-j| \geq 2; \\ &s_i^2 = 1 \ ; \\ &s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ , \quad i \geq 1 \ ; \\ &s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \ . \end{aligned}$$

The indexing in this definition differs from the usual one (namely, what we call $s_0, \ldots, s_{n-1}$ would have to be denoted by $s_n, \ldots, s_1$, respectively; cf. [B]). However, we find this labelling more convenient since it is respected by the natural embedding $B_n \hookrightarrow B_{n+1}$.

The elements of B_n can be thought of as *signed permutations*: a generator s_i, $i > 0$, swaps the i'th and $(i+1)$'st entries and the generator s_0 changes the sign of the first entry. As in any Coxeter group, the *length* $l(w)$ of an element w is the minimal number of generators whose product is w. Such a factorization of minimal length (or the corresponding sequence of indices) is called a *reduced decomposition*

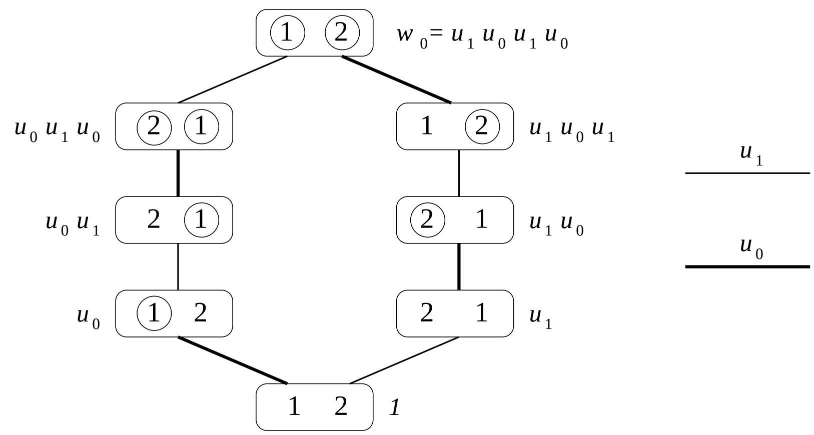

FIGURE A. The weak order of the hyperoctahedral group B_2

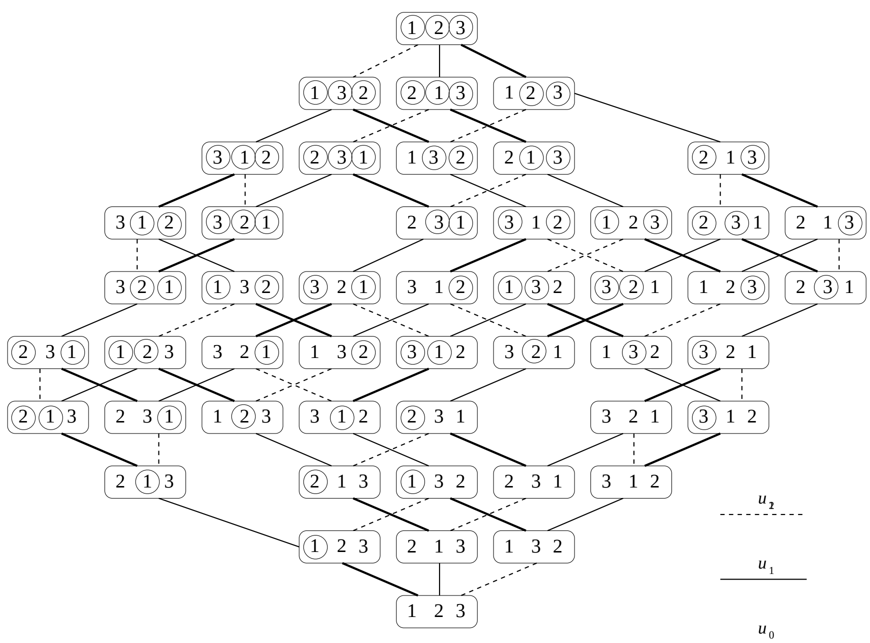

FIGURE B. The weak order of the hyperoctahedral group B_3

of w. The *weak order* on the group is defined as the transitive closure of the covering relation $ws_i \gtrdot w \Longleftrightarrow l(ws_i) > l(w)$.

The weak orders of the hyperoctahedral groups B_2 and B_3 are given in Figures A and B. To represent signed permutations, we circle their negative entries.

The hyperoctahedral group B_n acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ in the natural way. Namely, s_i interchanges x_i and x_{i+1} , for $i = 1, \ldots, n-1$, and the special generator s_0 acts by

$$s_0 f(x_1, x_2, \ldots) = f(-x_1, x_2, \ldots) \ .$$

The reversal of the indexing of generators has an annoying drawback: we have to change the sign in the usual definition of the divided differences. Rather than doing that (and creating a lot of confusion), we simply change the recurrence rule (1) into

$$(1') \qquad \partial_i X_{ws_i} = \begin{cases} -X_w & \text{if} \quad l(ws_i) = l(w) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

To include the special case $i = 0$ in $(1')$, we also define

$$\partial_0 f = \frac{f - s_0 f}{-x_1} \ ;$$

the negative sign in the denominator compensates for the one in $(1')$.

Thus one should replace condition (1) by $(1')$ in the formulations of the Problems 0-1-2-4, 0-2-3-4, and 0-1-3-4, while treating the B_n case.

2. Generalized configurations and the Yang-Baxter equation

The notion of a *generalized configuration* was introduced in [FK1]. It is a configuration of contiguous lines which cross a given vertical strip from left to right; each line is subdivided into "segments"; each segment has an associated variable. A configuration is assumed to be generic in the following sense: (i) no three lines intersect at the same point; (ii) no two lines intersect at an endpoint of any segment; (ii) no two intersection points lie on the same vertical line.

In the B_n case, this notion assumes an additional flavor. Namely, configurations are contained in a *semi-strip* bounded from below by a *bottom mirror* (cf. [Ch]). The lines of a configuration are allowed to touch the bottom; corresponding points are called *points of reflection.* Whenever this happens, an associated variable changes its sign. An example of a B_n-configuration is given on Figure 1.

Intersection points and points of reflection will be of a particular interest to us. Each intersection point has a *level number* which indicates how many lines there are below this point (the point itself contributes 1). For example, the intersection points on Figure 1 have level numbers (from left to right) 1, 1, 2, and 1. By definition, the level number of a point of reflection is 0.

Let $\mathcal{C}$ be a configuration of the described type. Order its intersection and reflection points altogether from left to right; then write down their level numbers. The resulting sequence of integers $a(\mathcal{C}) = a_1 a_2 \dots$ is called a *word associated with* $\mathcal{C}$. In our running example, $a(\mathcal{C}) = 101201$. Now it is time to bring the variables into the picture. Assume that $\mathcal{A}$ is an associative algebra and $\{h_i(x) \ : \ i = 0, 1, \dots\}$ is a family of elements of $\mathcal{A}$ which depend on a formal variable x (we always assume that the main field contains all participating formal variables as independent transcendentals). Then the *associated expression* for a configuration $\mathcal{C}$ is

$$\Phi(\mathcal{C}) = h_{a_1}(z_1) \ h_{a_2}(z_2) \ \dots$$

where, as before, $a_1 a_2 \dots$ is an associated word and z_i is one of the following: if $a_i = 0$, then z_i is the variable related to the corresponding point of reflection (to the left of it); if $a_i > 0$, then $z_i = x_i - y_i$ where x_i and y_i are the variables for the segments intersecting at the corresponding point, x_i being a variable for the segment which is above to the left of this point. In the example of Figure 1,

$$\Phi(\mathcal{C}) = \Phi(\mathcal{C}; x, y, z, u, v) = h_1(y - u) h_0(y) h_1(v + y) h_2(x + y) h_0(v) h_1(x + v) \ .$$

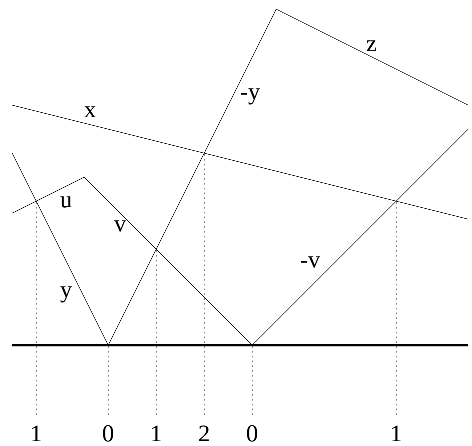

FIGURE 1

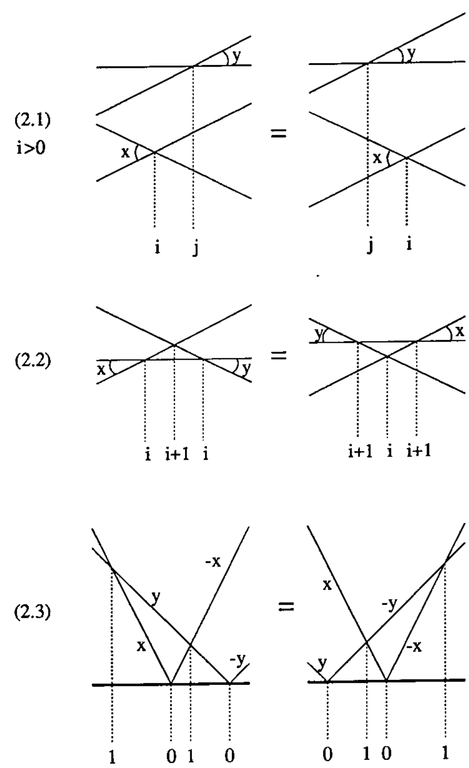

FIGURE 2

Informally, the variables associated with the segments are their "slopes"; an argument of each factor in $\Phi(\mathcal{C})$ is the corresponding "angle of intersection".

The *Yang-Baxter equations* (see, e.g., [Ch] and references therein) are certain conditions on the $h_i(x)$ which allow us to transform configurations without changing their associated expression. The type B YBE are

$$h_i(x)h_j(y) = h_j(y)h_i(x) \quad \text{if} \quad |i-j| \geq 2\ ; \tag{2.1}$$

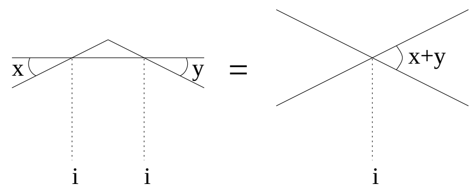

FIGURE 3

$$h_i(x)h_{i+1}(x+y)h_i(y) = h_{i+1}(y)h_i(x+y)h_{i+1}(x) \quad \text{if} \quad i \geq 1 \; ; \tag{2.2}$$

$$h_1(x-y)h_0(x)h_1(x+y)h_0(y) = h_0(y)h_1(x+y)h_0(x)h_1(x-y) \; . \tag{2.3}$$

Each of these equations has its pictorial interpretation; see Figure 2.

Following [FK3], we introduce the additional condition

$$h_i(x)h_i(y) = h_i(x+y) \; , \quad h_i(0) = 1 \quad \text{for} \quad i \geq 1 \tag{2.4}$$

(cf. Figure 3) which means that we are interested in *exponential solutions* of the YBE. Relations (2.1)-(2.4) have various nice implications which can be derived by braid manipulation that replaces cumbersome algebraic computations. Informally, algebraic identities can be proved by moving lines according to the rules of Figures 2 and 3.

3. Examples of solutions

There is a natural approach to constructing solutions of the equations (2.1)-(2.4). Assume that $\mathcal{A}$ is a *local* associative algebra (in the sense of [V]) with generators u_0, u_1, u_2, ... which means that

$$u_i u_j = u_j u_i \quad \text{if} \quad |i-j| \geq 2 \; . \tag{3.1}$$

Define $h_i(x)$ by

$$h_i(x) = \exp(x u_i) \; . \tag{3.2}$$

Then (2.1) and (2.4) are guaranteed, and we only need the Yang-Baxter equations (2.2)-(2.3) to be satisfied. Rewrite (2.2)-(2.3) as

$$e^{xu_i} e^{(x+y)u_{i+1}} e^{yu_i} = e^{yu_{i+1}} e^{(x+y)u_i} e^{xu_{i+1}} \tag{3.3}$$

and

$$[e^{xu_0} e^{xu_1} e^{xu_0}, e^{yu_0} e^{yu_1} e^{yu_0}] = 0 \tag{3.4}$$

where $[X, Y]$ denotes a commutator $XY - YX$. These equations were studied in [FK3] where the following solutions were suggested.

3.1 Example. *The nilCoxeter algebra of the hyperoctahedral group.* This is the algebra defined by

$$\begin{aligned} &u_i u_j = u_j u_i \; , \quad |i-j| \geq 2; \\ &u_i^2 = 0 \; ; \\ &u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \; , \quad i \geq 1 \; ; \\ &u_0 u_1 u_0 u_1 = u_1 u_0 u_1 u_0 \; . \end{aligned}$$

These relations are satisfied by the divided differences ∂_i; thus one can think of the nilCoxeter algebra as of the algebra of divided difference operators.

Example 3.1 will be the main one in this paper. The relation $u_i^2 = 0$ implies that (3.2) can be rewritten as $h_i(x) = 1 + xu_i$. Checking that these $h_i(x)$ satisfy the conditions (2.1)-(2.4) is straightforward.

The nilCoxeter algebra (of any Coxeter group W) has the following alternative description. For $w \in W$, take any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ and identify w with the element $u_{i_1} \cdots u_{i_l}$ of the nilCoxeter algebra. These elements form a linear basis of the nilCoxeter algebra, and the multiplication rule is

$$w \cdot v = \begin{cases} \text{usual product } \ wv \ \text{ if } \ l(w) + l(v) = l(wv), \\ 0 \ , \ \text{otherwise} \ . \end{cases}$$

In this paper, we will frequently make use of this description, while expanding various expressions in the nilCoxeter algebra of B_n in the basis of group elements.

3.2 Example. *Universal enveloping algebra of* $U_+(so\,(2n+1))$. This algebra can be defined as the local algebra with generators $u_1, u_2, \ldots$ subject to Serre relations

$$\begin{aligned} &[u_i, [u_i, u_{i\pm 1}]] = 0 \ , \quad i \geq 1 \ ; \\ &[u_0, [u_0, u_1]] = 0 \ ; \\ &[u_1, [u_1, [u_1, u_0]]] = 0 \ . \end{aligned}$$

As shown in [FK3], this universal enveloping algebra provides an exponential solution to the Yang-Baxter equation, that is, (3.3) and (3.4) are satisfied.

4. Symmetric expressions

By analogy with [FK1], we will show now that the basic relations (2.1)-(2.4) (or (3.1)-(3.4)) imply that certain configurations produce *symmetric* expressions in the corresponding variables. In what follows we assume that (2.1)-(2.4) are satisfied.

4.1 Theorem. *For the configuration* $\mathcal{C}$ *of Figure* 4*,*

$$\Phi(\mathcal{C}; x, y) = \Phi(\mathcal{C}; y, x) \ .$$

In other words, $\Phi(\mathcal{C})$ *is symmetric in* x *and* y*.*

This statement has the following straightforward reformulation.

4.2 Proposition. *Let*

$$B(x) = h_{n-1}(x) \cdots h_1(x) h_0(x) h_1(x) \cdots h_{n-1}(x) \ . \tag{4.1}$$

Then $B(x)$ *and* $B(y)$ *commute.*

Special case: $n = 2$. Then $B(x) = h_1(x)h_0(x)h_1(x)$. Now use (2.3) and (2.4) to show that

$$\begin{aligned} B(x)B(y) &= h_1(x)h_0(x)h_1(x)h_1(y)h_0(y)h_1(y) \\ &= h_1(x)h_0(x)h_1(x+y)h_0(y)h_1(y) \\ &= h_1(x)h_0(x)h_1(x+y)h_0(y)h_1(y-x)h_1(x) \\ &= h_1(x)h_1(y-x)h_0(y)h_1(x+y)h_0(x)h_1(x) \\ &= h_1(y)h_0(y)h_1(y+x)h_0(x)h_1(x) \\ &= B(y)B(x) \ . \end{aligned}$$

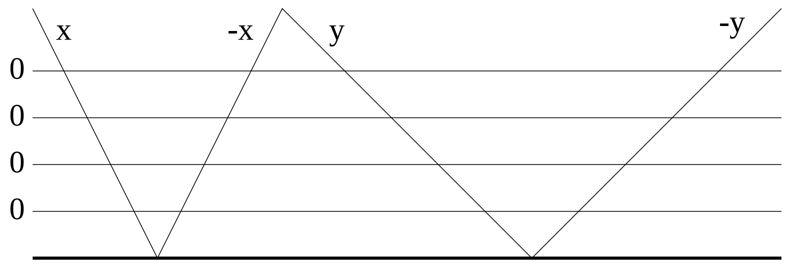

Figure 4

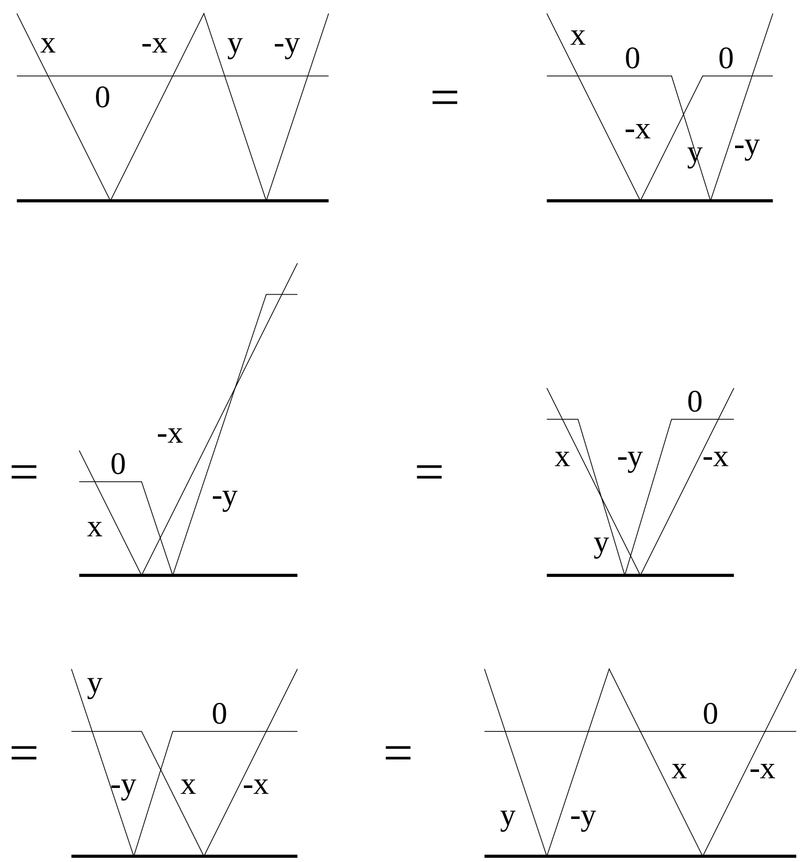

Figure 5

The same proof can be performed in the language of configurations — see Figure 5. Moreover, the geometric proof has the advantage of being easily adjustable for the general case of an arbitrary n.

Proof of Theorem 4.1 (*and Proposition* 4.2). Same transformations as in Figure 5, with additional horizontal lines added near the bottom mirror. □

4.3 Corollary. *Let*

$$H^{(n)}(x_1, x_2, \dots) = B(x_1)B(x_2)\cdots \tag{4.2}$$

where $B(x)$ is defined by (4.1)*, and the $h_i(x)$ satisfy the relations* (2.1)-(2.4)*. Then the expression $H^{(n)}(x_1, x_2, \dots)$ is symmetric in the x_i. Moreover, $H^{(n)}$ obeys the following cancellation rule:*

$$H^{(n)}(x_1, -x_1, x_2, , x_3, \dots) = H^{(n)}(x_2, x_3, \dots)\ . \tag{4.3}$$

Proof. The symmetry is an immediate consequence of Proposition 4.2. The cancellation rule follows from the identity $B(-x)B(x) = 1$. □

Corollary 4.3 suggests that one can construct non-trivial examples of symmetric functions by taking any solution of (2.1)-(2.4), then any representation of the corresponding algebra $\mathcal{A}$, then applying the operator representing $H^{(n)}(x_1, x_2, \dots)$ to any vector and, finally, taking any coordinate of the image. Not only will those functions be symmetric; by a theorem of Pragacz [P], the cancellation rule (4.3) implies that they will belong to the subring $\Omega(x_1, x_2, \dots)$ of the ring $\Lambda(x_1, x_2, \dots)$ of symmetric functions that is generated by odd power sums; equivalently, any such function is a linear combination of Schur P-functions.

Surprisingly enough, the expression $H^{(n)}(x_1, x_2, \dots)$ can be alternatively defined by a quite different configuration.

4.4 Theorem. *Let $\mathcal{C}$ be the configuration defined by Figure* 6*. Then*

$$\Phi(\mathcal{C}; x_1, \dots, x_n) = H^{(n)}(x_1, \dots, x_n)\ .$$

Proof. See Figure 7. □

Remark. If the number of generators is $m > n$, then, to get $H^{(m)}(x_1, \dots, x_n)$, one only needs to add $m-n$ horizontal lines near the bottom mirror in the configuration of Figure 6.

Theorem 4.4 allows us to relate the type B and type A constructions to each other. Note that the configuration of Figure 6 coincides with one of [FK1, Figure 14], up to renumbering the variables x_i in the opposite order (this is not essential since the expression is symmetric in the x_i), setting $y_i = -x_i$, and attaching the bottom mirror. Since Figure 14 of [FK1] defines the ordinary (i.e, type A) double stable Schubert expression $G(x_1, \dots, x_n; y_1, \dots, y_n)$, the above observation has the following precise formulation.

4.5 Theorem. *Let $\{h_i(x)\ :\ i = 1, \dots, n-1\}$ be any solution of* (2.1), (2.2)*, and* (2.4)*; in other words, let $\{h_i(x)\}$ be an exponential solution of the YBE of type A_{n-1}. Define $h_0(x) = 1$. Then* (2.3) *obviously holds and so $H^{(n)}$ is well-defined. Moreover, in this case*

$$H^{(n)}(x_1, \dots, x_n) = G(x_1, \dots, x_n; -x_1, \dots, -x_n)$$

where $G(\dots)$ is the double stable Schubert expression (*see* [FK1]).

In the special case of the nilCoxeter solution of Example 3.1 we obtain the B_n-analogues of the Stanley symmetric functions [S], or stable Schubert polynomials. These functions are studied in Section 8.

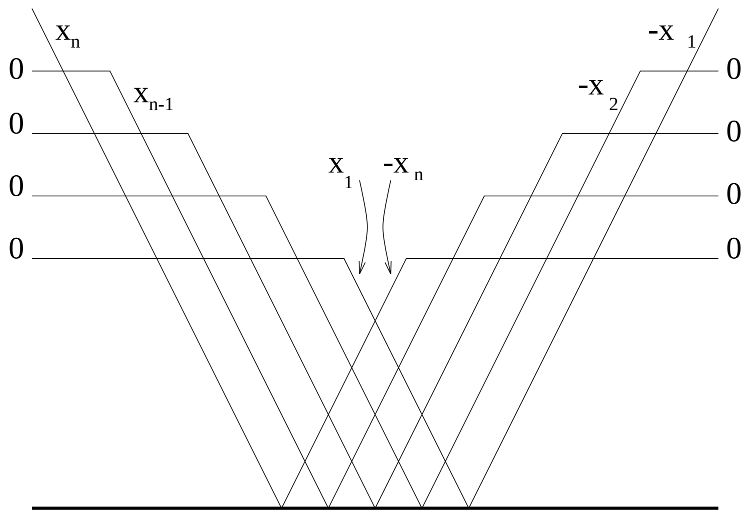

FIGURE 6

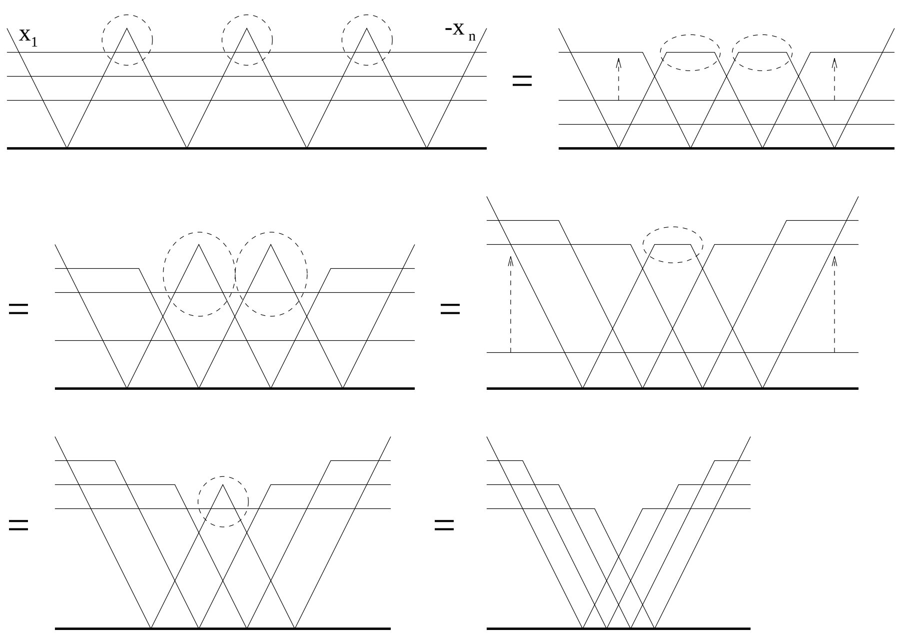

FIGURE 7

5. SCHUBERT EXPRESSIONS

Define the B_n-analogue of the generalized Schubert expression by

$$\mathfrak{b}^{(n)}(x_1, \dots, x_n) = H^{(n)}(x_1, \dots, x_n)\mathfrak{S}(-x_1, \dots, -x_{n-1}) \tag{5.1}$$

where $\mathfrak{S}(x_1,\ldots,x_{n-1})$ is the A_{n-1}-Schubert expression as defined in [FS], [FK1]. In other words,

$$\mathfrak{b}^{(n)}(x_1,\ldots,x_n)=B(x_1)\cdots B(x_n)A_1(-x_1)\cdots A_{n-1}(-x_{n-1}) \tag{5.2}$$

where

$$A_i(x)=h_{n-1}(x)h_{n-2}(x)\cdots h_i(x) \tag{5.3}$$

(recall that $B(x)$ is defined by (4.1)).

The formula (5.2) can be simplified.

5.1 Theorem. *$\mathfrak{b}^{(n)}(x_1,\ldots,x_n)$ is equal to the expression defined by Figure 8. In other words,*

(5.4)

$$\mathfrak{b}^{(n)}(x_1,\ldots,x_n)=\mathfrak{S}(x_n,x_{n-1},\ldots,x_2)\prod_{i=0}^{n-1}\left(h_0(x_{n-i})\prod_{j=1}^{n-i-1}h_j(x_{n-i-j}+x_{n-i})\right)$$

where, as before, $\mathfrak{S}(x_n,x_{n-1},\ldots,x_2)=A_1(x_n)\cdots A_{n-1}(x_2)$ and in the products $\prod\cdots$ the factors are multiplied left-to-right, according to the increase of i and j, respectively.

Note that the total number of factors $h_{\ldots}(\ldots)$ in (5.4) is n^2, the length of the longest element w_0 of the hyperoctahedral group B_n with n generators. Moreover, it can be immediately seen from Figure 8 that these factors are in a natural order-respecting bijection with the entries of the lexicographically maximal reduced decomposition of w_0:

$$n-1,\ n-2,\ \ldots,\ 2,\ 1,\ 0,\ n-1,\ n-2,\ \ldots,\ 2,\ 1,\ 0,\ \ldots\ ,\\ n-1,\ n-2,\ \ldots,\ 2,\ 1,\ 0\ .$$

5.2 Examples.

$$\begin{aligned}
&\mathfrak{b}^{(1)}(x_1)=h_0(x_1),\\
&\mathfrak{b}^{(2)}(x_1,x_2)=h_1(x_2)h_0(x_2)h_1(x_1+x_2)h_0(x_1),\\
&\mathfrak{b}^{(3)}(x_1,x_2,x_3)=h_2(x_3)h_1(x_3)h_2(x_2)h_0(x_3)h_1(x_2+x_3)\\
&\qquad\qquad\times h_2(x_1+x_3)h_0(x_2)h_1(x_1+x_2)h_0(x_1).
\end{aligned}$$

Proof of Theorem 5.1. Let Φ_6 and Φ_8 be the expressions defined by configurations of Figures 6 and 8, respectively. Then

$$\Phi_6=\Phi_8\tilde{A}_{n-1}(x_{n-1})\cdots\tilde{A}_2(x_2)\tilde{A}_1(x_1)$$

where

$$\tilde{A}_i(x)=h_i(x)h_{i+1}(x)\cdots h_{n-1}(x)\ .$$

Since (2.4) implies that $(\tilde{A}_i(x))^{-1}=A_i(-x)$, it follows from Theorem 4.4 that

$$\begin{aligned}
\Phi_8&=\Phi_6A_1(-x_1)A_2(-x_2)\cdots A_{n-1}(-x_{n-1})\\
&=H^{(n)}(x_1,\ldots,x_n)\mathfrak{S}(-x_1,\ldots,-x_{n-1})\\
&=\mathfrak{b}^{(n)}(x_1,\ldots,x_n). \qquad\square
\end{aligned}$$

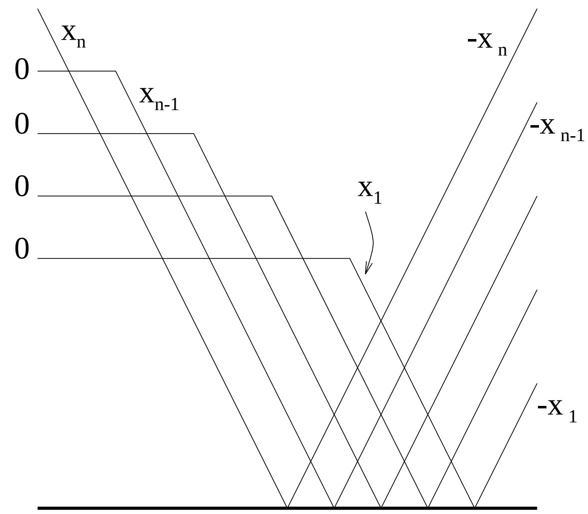

FIGURE 8

6. SCHUBERT POLYNOMIALS OF THE FIRST KIND

In the rest of this paper we study the main example of solution of (2.1)-(2.4), namely, the one related to the nilCoxeter algebra of the hyperoctahedral group (Example 3.1). In this example, $h_i(x) = 1 + xu_i$ where u_i is the i'th generator. By analogy with the case of the symmetric group (cf. [FK1]), we define the *type B Schubert polynomials of the first kind* by expanding the corresponding expression in the nilCoxeter algebra in the basis of group elements:

$$\mathfrak{b}^{(n)}(x_1, \ldots, x_n) = \sum_{w \in B_n} \mathfrak{b}_w^{(n)}(x_1, \ldots, x_n)\ w\ . \tag{6.1}$$

6.1 Examples. (Cf. Examples 5.2.) In B_1,

$$\mathfrak{b}^{(1)}(x_1) = 1 + x_1 u_0$$

and therefore $\mathfrak{b}_1^{(1)} = 1$ and $\mathfrak{b}_{u_0}^{(1)} = x_1$.

In B_2,

$$\mathfrak{b}^{(2)}(x_1, x_2) = (1 + x_2 u_1)(1 + x_2 u_0)(1 + (x_1 + x_2)u_1)(1 + x_1 u_0)\ .$$

Expanding in the basis of group elements, we obtain the B_2-Schubert polynomials of the first kind

w	$\mathfrak{b}_w^{(2)}$
1	1
u_0	$x_1 + x_2$
u_1	$x_1 + 2x_2$
$u_0 u_1$	$x_2(x_1 + x_2)$
$u_1 u_0$	$(x_1 + x_2)^2$
$u_0 u_1 u_0$	$x_1 x_2 (x_1 + x_2)$
$u_1 u_0 u_1$	$x_2^2 (x_1 + x_2)$
w_0	$x_1 x_2^2 (x_1 + x_2)$

In B_3,

$$\mathfrak{b}^{(3)}(x_1,x_2,x_3) = (1+x_3u_2)(1+x_3u_1)(1+x_2u_2)(1+x_3u_0)(1+(x_2+x_3)u_1) \times (1+(x_1+x_3)u_2)(1+x_2u_0)(1+(x_1+x_2)u_1)(1+x_1u_0).$$

Expanding the right-hand side, we obtain the table of the polynomials $\mathfrak{b}_w^{(3)}$ given in Figure 9.

It immediately follows from the definitions that, in general, the top polynomial of the first kind is given by

$$\mathfrak{b}_{w_0}^{(n)}(x_1,\dots,x_n) = \prod_{k=1}^{n}(x_k)^k \prod_{1\le i<j\le n}(x_i+x_j)$$

(cf. (5)).

By analogy with (6.1), let us define the *Stanley polynomials of type B* by

$$H^{(n)}(x_1,\dots,x_k) = \sum_{w\in B_n} H_w^{(n)}(x_1,\dots,x_k)\ w\ ; \tag{6.2}$$

here, as before, w is identified with the corresponding product of generators of the nilCoxeter algebra. For example,

$$H^{(2)}(x_1,x_2) = (1+x_2u_1)(1+x_2u_0)(1+(x_1+x_2)u_1)(1+x_1u_0)(1+x_1u_1)$$

and thus, e.g., $H_{w_0}^{(2)}(x_1,x_2) = x_1x_2(x_1+x_2)^2$. (Note that, in general, k and n in (6.2) need not be equal.) As shown in Section 4 (see Corollary 4.3 and the paragraph immediately following its proof), these functions are indeed symmetric, and the following result holds.

6.2 Lemma. *The Stanley polynomials $H_w^{(n)}$ of type B belong to the ring Ω generated by odd power sums. They are integer linear combinations of Schur P-functions.* □

We discuss these polynomials in greater detail in Section 8.

The definitions (6.1) and (6.2) can be straightforwardly restated in terms of reduced decompositions and "compatible sequences". Use (4.1)-(4.2) to rewrite (6.2) as

$$H_w^{(n)}(x_1,\dots,x_k) = \sum_{a_1,\dots,a_l\in R(w)} \sum_{\substack{1\le b_1\le\cdots\le b_l\le k\\ a_i<a_{i+1}>a_{i+2}\implies b_i<b_{i+2}}} 2^{\gamma(\mathbf{a},\mathbf{b})} x_{b_1}x_{b_2}\cdots x_{b_l} \tag{6.3}$$

where $R(w)$ is the set of reduced decompositions of w and

$$\gamma(\mathbf{a},\mathbf{b}) = \#\{b_i\} - \#\{i\ :\ a_i=0\}$$

(here $\#\{b_i\}$ denotes the number of *different* entries in the sequence $b_1, \dots ,b_l$). Correspondingly, (5.1) can be presented as

$$\mathfrak{b}_w^{(n)}(x_1,\dots,x_n) = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in A_{n-1}}} H_u^{(n)}(x_1,\dots,x_n)\mathfrak{S}_v(-x_1,\dots,-x_{n-1}) \tag{6.4}$$

where $\mathfrak{S}_v$ is the ordinary Schubert polynomial for the symmetric group $A_{n-1}=S_n$. It is also possible to entirely rewrite the definition of Theorem 5.1 in terms of reduced decompositions and compatible sequences. We avoid doing so since the

w		$\lvert R(w)\rvert$	$\mathfrak{b}_w^{(3)}$
123	1	1	1
$\overline{1}23$	u_0	1	$x_1+x_2+x_3$
213	u_1	1	$x_1+2x_2+2x_3$
132	u_2	1	$x_1+x_2+2x_3$
$2\overline{1}3$	u_0u_1	1	$(x_2+x_3)(x_1+x_2+x_3)$
$\overline{1}32$	u_0u_2	2	$(x_1+x_2+2x_3)(x_1+x_2+x_3)$
$\overline{2}13$	u_1u_0	1	$(x_1+x_2+x_3)^2$
231	u_1u_2	1	$x_1x_2+2x_1x_3+2x_2x_3+2x_3^2$
312	u_2u_1	1	$(x_1+x_2+x_3)^2+(x_2+x_3)^2$
$\overline{2}\overline{1}3$	$u_0u_1u_0$	1	$(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$23\overline{1}$	$u_0u_1u_2$	1	$x_3(x_1+x_3)(x_2+x_3)$
$3\overline{1}2$	$u_0u_2u_1$	2	$(x_2+x_3)((x_1+x_2+x_3)^2+x_2x_3)$
$1\overline{2}3$	$u_1u_0u_1$	1	$(x_2+x_3)(x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2)$
$\overline{2}31$	$u_1u_0u_2$	2	$x_1^2x_2+x_1x_2^2+2x_1^2x_3+4x_1x_2x_3+3x_1x_3^2$ $+2x_2^2x_3+3x_2x_3^2+x_3^3$
321	$u_1u_2u_1$	2	$x_1^2x_2+2x_1x_2^2+2x_1^2x_3+4x_1x_3^2+4x_2^2x_3$ $+6x_2x_3^2+2x_3^3+6x_1x_2x_3$
$\overline{3}12$	$u_2u_1u_0$	1	$(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3)$ $+x_1x_2x_3$
$\overline{1}\overline{2}3$	$u_0u_1u_0u_1$	2	$(x_1+x_2)(x_1+x_3)(x_2+x_3)^2$
$\overline{2}3\overline{1}$	$u_0u_1u_0u_2$	2	$x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$32\overline{1}$	$u_0u_1u_2u_1$	3	$x_3(x_2+x_3)(x_1+x_3)(x_1+2x_2+x_3)$
$\overline{3}\overline{1}2$	$u_0u_2u_1u_0$	2	$(x_1+x_2)(x_1+x_3)(x_2+x_3)(x_1+x_2+x_3)$
$13\overline{2}$	$u_1u_0u_1u_2$	1	$x_3^2(x_2+x_3)(x_1+x_3)$
$3\overline{2}1$	$u_1u_0u_2u_1$	2	$(x_2+x_3)(x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2$ $+2x_2^2x_3+2x_2x_3^2+3x_1x_2x_3)$
$\overline{3}21$	$u_1u_2u_1u_0$	3	$3x_1x_3^3+3x_2x_3^3+x_3^4+x_1^3x_2+2x_1^2x_2^2+6x_1^2x_2x_3$ $+2x_1^3x_3+4x_1^2x_3^2+6x_1x_2^2x_3+8x_1x_2x_3^2+x_1x_2^3$ $+2x_2^3x_3+4x_2^2x_3^2$
$1\overline{3}2$	$u_2u_1u_0u_1$	1	$(x_2+x_3)((x_2^2+x_3^2)(x_1+x_2+x_3)+x_1x_2x_3)$
$\overline{1}3\overline{2}$	$u_0u_1u_0u_1u_2$	3	$x_3^2(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$3\overline{2}\overline{1}$	$u_0u_1u_0u_2u_1$	2	$x_2x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{3}2\overline{1}$	$u_0u_1u_2u_1u_0$	5	$x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)(x_1+x_2+x_3)$
$\overline{1}\overline{3}2$	$u_0u_2u_1u_0u_1$	3	$(x_1+x_2)(x_1+x_3)(x_2+x_3)(x_2^2+x_2x_3+x_3^2)$
$31\overline{2}$	$u_1u_0u_1u_2u_1$	3	$x_3^2(x_1+2x_2)(x_1+x_3)(x_2+x_3)$
$\overline{3}\overline{2}1$	$u_1u_0u_2u_1u_0$	2	$(x_1+x_2)(x_1+x_3)(x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)$
$2\overline{3}1$	$u_1u_2u_1u_0u_1$	3	$6x_1x_2^2x_3^2+4x_2^3x_3^2+4x_2^2x_3^3+2x_2^4x_3+4x_1x_2^3x_3$ $+4x_1x_2x_3^3+2x_2x_3^4+x_1^2x_2^3+2x_1^2x_2^2x_3+2x_1^2x_2x_3^2$ $+x_1^2x_3^3+x_1x_3^4+x_1x_2^4$
$12\overline{3}$	$u_2u_1u_0u_1u_2$	1	$x_3^3(x_2+x_3)(x_1+x_3)$
$3\overline{1}\overline{2}$	$u_0u_1u_0u_1u_2u_1$	5	$x_2x_3^2(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{3}\overline{2}\overline{1}$	$u_0u_1u_0u_2u_1u_0$	2	$x_1x_2x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$2\overline{3}\overline{1}$	$u_0u_1u_2u_1u_0u_1$	5	$x_2x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)^2$
$\overline{1}2\overline{3}$	$u_0u_2u_1u_0u_1u_2$	4	$x_3^3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{3}1\overline{2}$	$u_1u_0u_1u_2u_1u_0$	5	$x_3^2(x_1+x_2)^2(x_1+x_3)(x_2+x_3)$
$\overline{2}\overline{3}1$	$u_1u_0u_2u_1u_0u_1$	5	$(x_1x_2^2+x_1x_3^2+x_1x_2x_3+x_2^2x_3+x_2x_3^2)$ $(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$21\overline{3}$	$u_1u_2u_1u_0u_1u_2$	4	$x_3^3(x_1+2x_2)(x_1+x_3)(x_2+x_3)$
$\overline{3}\overline{1}\overline{2}$	$u_0u_1u_0u_1u_2u_1u_0$	7	$x_1x_2x_3^2(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{2}\overline{3}\overline{1}$	$u_0u_1u_0u_2u_1u_0u_1$	7	$x_1x_2x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)^2$
$2\overline{1}\overline{3}$	$u_0u_1u_2u_1u_0u_1u_2$	9	$x_2x_3^3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$1\overline{3}\overline{2}$	$u_1u_0u_1u_2u_1u_0u_1$	5	$x_2^2x_3^2(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{2}1\overline{3}$	$u_1u_0u_2u_1u_0u_1u_2$	9	$x_3^3(x_1+x_2)^2(x_1+x_3)(x_2+x_3)$
$\overline{1}\overline{3}\overline{2}$	$u_0u_1u_0u_1u_2u_1u_0u_1$	12	$x_1x_2^2x_3^2(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{2}\overline{1}\overline{3}$	$u_0u_1u_0u_2u_1u_0u_1u_2$	16	$x_1x_2x_3^3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$1\overline{2}\overline{3}$	$u_1u_0u_1u_2u_1u_0u_1u_2$	14	$x_2^2x_3^3(x_1+x_2)(x_1+x_3)(x_2+x_3)$
$\overline{1}\overline{2}\overline{3}$	$u_0u_1u_0u_1u_2u_1u_0u_1u_2$	42	$x_1x_2^2x_3^3(x_1+x_2)(x_1+x_3)(x_2+x_3)$

FIGURE 9. Type B_3 Schubert polynomials of the first kind

resulting formulas are rather messy; we also think that the following geometric approach (cf. [FK1, Section 6]) is more natural.

Both $\mathfrak{b}_w^{(n)}$ and $H_w^{(n)}$ have a direct combinatorial interpretation in terms of "resolved configurations"; this interpretation can actually be applied to a family of polynomials which come from any configuration $\mathcal{C}$. Take all the intersection points of $\mathcal{C}$ and "resolve" each of them either as $\times$ or as $\asymp$. Then take all points of reflection and resolve each of them either as $\smile$ or as $\vee$; the latter corresponds to changing a "sign", or a "spin", of the corresponding string. If a configuration has N intersection and reflection points altogether, then there are 2^N ways of producing such a resolution. Each of the 2^N resolved configurations is a "signed braid" which naturally gives an element w of the hyperoctahedral group. Reading the $\times$- and $\vee$-points from left to right produces a decomposition of w into a product of generators. Let $\mathcal{C}_w$, for a given w, denote the set of resolved configurations which give w and for which this decomposition is *reduced*. Then the polynomials Φ_w associated with $\mathcal{C}$, that is,

$$\Phi(\mathcal{C}; x_1, x_2, \dots) = \sum_w \Phi_w(x_1, x_2, \dots)\ w,$$

can be expressed as

$$\Phi_w(x_1, x_2, \dots) = \sum_{c \in \mathcal{C}_w} \Big(\Big(\prod (x_i - x_j)\Big)\Big(\prod x_k\Big)\Big) \tag{6.5}$$

where the first product is taken over all intersections in $\mathcal{C}$ and the second one — over all "change-of-sign" (i.e., $\vee$-) points.

This interpretation enables us to prove the *stability* of $\mathfrak{b}_w^{(n)}$ and $H_w^{(n)}$.

6.3 Theorem. *Let B_n and B_m, $n < m$, be the hyperoctahedral groups with generators $s_0, \dots, s_{n-1}$ and $s_0, \dots, s_{m-1}$, respectively. Then, for any $w \in B_n \subset B_m$,*

$$\mathfrak{b}_w^{(n)}(x_1, \dots, x_n) = \mathfrak{b}_w^{(m)}(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{m-n}) \tag{6.6}$$

and

$$H_w^{(n)}(x_1, \dots, x_k) = H_w^{(m)}(x_1, \dots, x_k)\ . \tag{6.7}$$

In other words, the $H_w^{(n)}$ do not depend on the superscript n (so we may drop it), and the $\mathfrak{b}_w^{(n)}$ are stable in the weaker sense of (4): the coefficient of any monomial in $\mathfrak{b}_w^{(n)}(x_1, \dots, x_n)$ stabilizes as $n \to \infty$. Thus we can introduce a well-defined formal power series

(6.8)
$$\mathfrak{b}_w(x_1, x_2, \dots) = \lim_{n\to\infty} \mathfrak{b}_w^{(n)}(x_1, \dots, x_n) = H^{(n)}(x_1, x_2, \dots)\mathfrak{S}(-x_1, \dots, -x_{n-1})$$

which could be viewed as a limiting form of the type B Schubert polynomial of the first kind. Here $H^{(n)}(x_1, x_2, \dots)$ is a symmetric expression in infinitely many variables $x_1, x_2, \dots$.

Proof of Theorem 6.3. Let $I_{>n}$ denote the two-sided ideal in the nilCoxeter algebra that is generated by $u_{n+1}, u_{n+2}, \ldots$ Then (6.6) and (6.7) can be restated, respectively, as

$$\mathfrak{b}^{(n)}(x_1, \ldots, x_n) \equiv \mathfrak{b}^{(m)}(x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{m-n}) \mod I_{>n}$$

and

$$H^{(n)}(x_1, \ldots, x_k) \equiv H^{(m)}(x_1, \ldots, x_k) \mod I_{>n}$$

for $m > n$. The latter congruence is immediate from the definition of $H^{(n)}$ (one can also interpret it geometrically; cf. Figure 4). As to the former one, note that the stability of the ordinary (type A) Schubert polynomials can be reformulated as

$$\mathfrak{S}^{(n)}(x_1, \ldots, x_n) \equiv \mathfrak{S}^{(m)}(x_1, \ldots, x_m) \mod I_{>n} \ ,$$

implying

$$\begin{aligned} \mathfrak{b}^{(n)}(x_1, \ldots, x_n) &= H^{(n)}(x_1, \ldots, x_n)\mathfrak{S}^{(n)}(-x_1, \ldots, -x_{n-1}) \\ &\equiv H^{(m)}(x_1, \ldots, x_n)\mathfrak{S}^{(m)}(-x_1, \ldots, -x_{n-1}, 0, \ldots, 0) \mod I_{>n} \\ &= H^{(m)}(x_1, \ldots, x_n, 0, \ldots, 0)\mathfrak{S}^{(m)}(-x_1, \ldots, -x_{n-1}, 0, \ldots, 0) \\ &= \mathfrak{b}^{(m)}(x_1, \ldots, x_n, 0, \ldots, 0). \qquad \square \end{aligned}$$

Recall that the divided difference operator ∂_i is defined by

$$\partial_i f(x_1, \ldots) = \frac{f(x_1, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}} \ . \tag{6.9}$$

We will show now that the Schubert polynomials of the first kind satisfy the divided difference recurrences (1′) for all $i \geq 1$.

6.4 Theorem. *For any $w \in B_n$ and $i \geq 1$,*

$$-\partial_i \, \mathfrak{b}_{ws_i}^{(n)} \ = \ \begin{cases} \mathfrak{b}_w^{(n)} & \text{if} \quad l(ws_i) = l(w) + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.10}$$

(Unfortunately, (6.10) is false for $i = 0$. Otherwise the polynomials $\mathfrak{b}_w^{(n)}$ would satisfy the conditions (0)-(4) of the introduction, which is impossible.)

Proof. As before, let u_i be the generators of the nilCoxeter algebra. Then the theorem is equivalent to the identity

$$-\partial_i \ \sum_w \mathfrak{b}_{ws_i}^{(n)} \cdot (ws_i) = \sum_w \mathfrak{b}_w^{(n)} w u_i \ \ , \tag{6.11}$$

where in the left-hand side w is interpreted as an element of the symmetric group, and in the right-hand side w is identified with the basis element of the nilCoxeter algebra. In turn, (6.11) can be rewritten as

$$-\partial_i \mathfrak{b}^{(n)} = \mathfrak{b}^{(n)} u_i \quad .$$

Now recall that $H^{(n)}$ is symmetric in the x_i and therefore

$$-\partial_i \mathfrak{b}^{(n)} = -H^{(n)}(x_1, \ldots) \, \partial_i \mathfrak{S}(-x_1, \ldots) = H^{(n)}(x_1, \ldots)\mathfrak{S}(-x_1, \ldots)u_i = \mathfrak{b}^{(n)} u_i \ .$$

In this computation, we used (5.1) and the identity $\partial_i \mathfrak{S} = \mathfrak{S} u_i$ (see [FS, Lemma 3.5]) which is just another way of stating the divided-differences recurrence for the ordinary Schubert polynomials. □

By taking limits, one can obtain an analogue of Theorem 6.4 for the power series $\mathfrak{b}_w$ defined in (6.8).

A proof of property (2) for the Schubert polynomials of the first kind will be given in Section 7.

7. Schubert polynomials of the second kind

The Schubert expression of the second kind is defined in the nilCoxeter algebra of the hyperoctahedral group by the formula

$$\mathfrak{B}^{(n)}(x_1,\dots,x_n) = \sqrt{H^{(n)}(x_1,\dots,x_n)}\ \mathfrak{S}(-x_1,\dots,-x_{n-1}) \tag{7.1}$$

(cf. (5.1)). In this formula, $H^{(n)}(x_1,\dots,x_n)$ has constant term 1, so we interpret the square root via the expansion $\sqrt{1+\alpha} = 1 + \frac{\alpha}{2} - \frac{\alpha^2}{8} + \dots$. Note that in our case this expansion is *finite* since every noncommutative monomial of degree $> n^2$ in the nilCoxeter algebra of B_n vanishes. We then define the *Schubert polynomials of the second kind* by expanding $\mathfrak{B}^{(n)}$ in the basis of the nilCoxeter algebra formed by the group elements:

$$\mathfrak{B}^{(n)}(x_1,\dots,x_n) = \sum_{w\in B_n} \mathfrak{B}_w^{(n)}(x_1,\dots,x_n)\ w\ ; \tag{7.2}$$

(cf. (6.1)). For example, the B_2-Schubert polynomials of the second kind can be obtained by expanding the expression

$$\begin{aligned}\mathfrak{B}^{(2)}(x_1,x_2) = &\sqrt{(1+x_2u_1)(1+x_2u_0)(1+(x_1+x_2)u_1)(1+x_1u_0)(1+x_1u_1)}\\ &\cdot (1-x_1u_1)\ .\end{aligned}$$

7.1 Theorem. *The Schubert polynomials of the second kind satisfy the recurrence relations* (1′)*:*

$$\partial_i \mathfrak{B}_{ws_i}^{(n)} = \begin{cases} -\mathfrak{B}_w^{(n)} & \textit{if} \quad l(ws_i) = l(w)+1, \\ 0 & \textit{otherwise} \end{cases} \tag{7.3}$$

for $i = 0,1,2,\dots$.

Proof. The verification of (7.3) for $i \geq 1$ is exactly the same as in the proof of Theorem 6.4, only replacing $H^{(n)}$ by $\sqrt{H^{(n)}}$. As to $i = 0$, we obtain, using Proposition 4.2 and (5.3),

$$
\begin{aligned}
-\partial_0\mathfrak{B}^{(n)} &= \frac{1}{x_1}\Big(\sqrt{B(x_1)B(x_2)\cdots}\ \mathfrak{S}(-x_1,-x_2,\dots)\\
&\qquad -\sqrt{B(-x_1)B(x_2)\cdots}\ \mathfrak{S}(x_1,-x_2,\dots)\Big)\\
&= \frac{1}{x_1}\sqrt{B(x_1)B(x_2)B(x_3)\cdots}\big(A_1(-x_1)-B(-x_1)A_1(x_1)\big)A_2(-x_2)A_3(-x_3)\cdots\\
&= \frac{1}{x_1}\sqrt{B(x_1)B(x_2)\cdots}\ A_1(-x_1)\big(1-h_0(-x_1)\big)A_2(-x_2)A_3(-x_3)\cdots\\
&= \frac{1}{x_1}\sqrt{B(x_1)B(x_2)\cdots}\ A_1(-x_1)\,x_1\,u_0\,A_2(-x_2)A_3(-x_3)\cdots\\
&= \sqrt{B(x_1)B(x_2)\cdots}\ A_1(-x_1)A_2(-x_2)A_3(-x_3)\cdots u_0\\
&= \mathfrak{B}^{(n)}u_0\ . \qquad \square
\end{aligned}
$$

7.2 Theorem. *The Schubert polynomials of the second kind satisfy the stability condition* (4)*:*

$$
\mathfrak{B}_w^{(n)}(x_1,\dots,x_n)=\mathfrak{B}_w^{(m)}(x_1,\dots,x_n,\underbrace{0,\dots,0}_{m-n}) \tag{7.4}
$$

for $i=0,1,2,\dots$.

Proof. The proof duplicates that of Theorem 6.3, with $\sqrt{H^{(n)}}$ instead of $H^{(n)}$. $\square$

Analogously to (6.8), the stability of the Schubert polynomials of the second kind allows us to introduce their stable limits, i.e., the power series

$$
\begin{aligned}
\mathfrak{B}_w(x_1,x_2,\dots) &= \lim_{n\to\infty}\mathfrak{B}_w^{(n)}(x_1,\dots,x_n)\\
&= \sqrt{H(x_1,x_2,\dots)}\ \mathfrak{S}(-x_1,\dots,-x_{n-1})
\end{aligned} \tag{7.5}
$$

in infinitely many variables x_1,x_2, $\dots$. These power series were first defined (in a different way) by Billey and Haiman [BH]. We will soon demonstrate the equivalence of the two definitions of the $\mathfrak{B}_w$.

The next theorem directly relates the symmetric power series $\sqrt{H}$ and H to each other; this relation will enable us to establish a connection between the Schubert polynomials of the two kinds.

7.3 Theorem. *Assume that the variables* x_1,x_2, $\dots$ *and* t_1,t_2, $\dots$ *are related by*

$$
\frac{p_k(x_1,x_2,\dots)}{2}=p_k(t_1,t_2,\dots),\quad k=1,3,5,\dots, \tag{7.6}
$$

where $p_k(x_1,x_2,\dots)=\sum x_i^k$ *and* $p_k(t_1,t_2,\dots)=\sum t_i^k$ *are* (*odd*) *power sums. Then*

$$
\sqrt{H(x_1,x_2,\dots)}=H(t_1,t_2,\dots)\ . \tag{7.7}
$$

Proof.

$$\begin{aligned}\log\left(\sqrt{H(x_1,x_2,\dots)}\right) &= \frac{1}{2}\log\left(H(x_1,x_2,\dots)\right) = \frac{1}{2}\log\Big(\prod_i B(x_i)\Big)\\ &= \frac{1}{2}\sum_i \log\left(B(x_i)\right) = \sum_i \log\left(B(t_i)\right)\\ &= \log\Big(\prod_i B(t_i)\Big) = \log\left(H(t_1,t_2,\dots)\right). \qquad \square\end{aligned}$$

Note that, in view of Lemma 6.1, the relation (7.6) can be regarded as a "change of variables". A λ-ring substitution essentially equivalent to (7.6) was first used by Billey and Haiman [BH] in their alternative definition of the power series $\mathfrak{B}_w$. In [BH], the $\mathfrak{B}_w$ are defined, in our current notation, by

$$\sum_w \mathfrak{B}_w(x_1,x_2,\dots)\, w \;=\; H(t_1,t_2,\dots)\mathfrak{S}(-x_1,-x_2,\dots) \tag{7.8}$$

where the x_i and the t_j are related to each other via (7.6). By virtue of (7.7), this definition is equivalent to our formula (7.5). (To be precise, the definition of $\mathfrak{B}_w$ in [BH] differs from ours in sign. Denoting their polynomials by $\mathfrak{B}_w^{\mathrm{BH}}$ to avoid confusion, we get

$$\mathfrak{B}_w^{\mathrm{BH}} = (-1)^{l(w)}\mathfrak{B}_w\ . \tag{7.9}$$

We chose our definition to make it consistent with the recurrence relations (1′) whereas the definition of [BH] respects (1).)

7.4 Relations between the type B Schubert polynomials of the two kinds. One can directly compute the type B Schubert polynomials of the first kind from their counterparts of the second kind, and vice versa, using the following application of the substitution (7.6). Suppose we know a polynomial $\mathfrak{b}_w(t_1,t_2,\dots)$ of the first kind. Expand it in the basis of type A_{n-1} Schubert polynomials, the coefficients being symmetric functions in the t_i:

$$\mathfrak{b}_w(t_1,t_2,\dots) = \sum_{v\in S_n} \alpha_v(t_1,t_2,\dots)\,\mathfrak{S}_v(t_1,t_2,\dots)\ . \tag{7.10}$$

(Such an expansion is unique and can be found, e.g., by a repeated use of divided differences.) The symmetric functions α_v will necessarily belong to the ring Ω generated by the odd power sums p_k. Express each $\alpha_v(t_1,t_2,\dots)$ in terms of the $p_k(t_1,t_2,\dots)$ and apply the substitution (7.6), thus obtaining the functions $\gamma_v(x_1,x_2,\dots) = \alpha_v(t_1,t_2,\dots)$. The polynomial of the second kind is now given by

$$\mathfrak{B}_w(x_1,x_2,\dots) = \sum_v \gamma_v(x_1,x_2,\dots)\,\mathfrak{S}_v(x_1,x_2,\dots)\ . \tag{7.11}$$

To compute $\mathfrak{b}_w$ from $\mathfrak{B}_w$, one can use the same algorithm with the inverse substitution.

To give an example, let us compute $\mathfrak{B}_{w_0}$ in B_2. We know (see Example 6.1) that $\mathfrak{b}_{w_0}^{(2)}(t_1,t_2) = t_1t_2^2(t_1+t_2)$. This can be uniquely expanded in the ordinary A_1-Schubert polynomials, which are 1 and t_1, with symmetric coefficients. The expansion is

$$\mathfrak{b}_{w_0}^{(2)}(t_1,t_2) = t_1t_2^2(t_1+t_2) = t_1t_2(t_1+t_2)^2\cdot 1 - t_1t_2(t_1+t_2)\cdot t_1\ .$$

Denoting $p_1 = t_1 + t_2$ and $p_3 = t_1^3 + t_2^3$, we obtain

$$\alpha_1(t_1, t_2) = t_1 t_2 (t_1 + t_2)^2 = p_1(p_1^3 - p_3)/3$$

and

$$\alpha_{u_1}(t_1, t_2) = -t_1 t_2 (t_1 + t_2) = (-p_1^3 + p_3)/3 \ .$$

Now plug in $p_1 = (x_1 + x_2)/2$ and $p_3 = (x_1^3 + x_2^3)/2$ (cf. (7.6)) to get

$$\gamma_1(x_1, x_2) = \frac{1}{3} \cdot \frac{x_1 + x_2}{2} \cdot \left(\left(\frac{x_1 + x_2}{2} \right)^3 - \frac{x_1^3 + x_2^3}{2} \right) = -\frac{1}{16} (x_1 - x_2)^2 (x_1 + x_2)^2$$

and

$$\gamma_{u_1}(x_1, x_2) = \frac{1}{3} \cdot \left(- \left(\frac{x_1 + x_2}{2} \right)^3 + \frac{x_1^3 + x_2^3}{2} \right) = \frac{1}{8} (x_1 - x_2)^2 (x_1 + x_2) \ .$$

Finally, use (7.11) to compute

$$\begin{aligned} \mathfrak{B}^{(2)}_{w_0}(x_1, x_2) &= -(x_1 - x_2)^2 (x_1 + x_2)^2/16 + x_1 \cdot (x_1 - x_2)^2 (x_1 + x_2)/8 \\ &= (x_1 - x_2)^3 (x_1 + x_2)/16 \ . \end{aligned}$$

The rest of the B_2-Schubert polynomials of the second kind can be computed from the divided difference recurrence relations (1′), producing the following table:

w	$\mathfrak{B}^{(2)}_w$
1	1
u_0	$(x_1 + x_2)/2$
u_1	x_2
$u_0 u_1$	$-(x_1 - x_2)(x_1 + x_2)/4$
$u_1 u_0$	$(x_1 + x_2)^2/4$
$u_0 u_1 u_0$	$-(x_1 - x_2)^2(x_1 + x_2)/8$
$u_1 u_0 u_1$	$-x_2(x_1 - x_2)(x_1 + x_2)/4$
w_0	$(x_1 - x_2)^3(x_1 + x_2)/16$

The above algorithm allows us to avoid calculations based on the expansion of the square root in (7.1). However, the λ-ring substitution (7.6) becomes progressively harder to compute, as n increases, so the computational advantages of this approach are questionable. Another problem that we hid while treating the B_2 case (fortunately, it did not affect our computations) is that the very definition of the substitution (7.6) is ambiguous in the case of finitely many variables, since the p_k are not algebraically independent. To resolve this problem, one can use the stability property (7.4): increase the number of variables n (thus going to a hyperoctahedral group B_n of a larger order) while keeping w fixed. The maximal degree of a coefficient α_v in (7.10) is $l = l(w)$. Therefore a representation of α_v as a polynomial in the p_k may only involve first $\left\lfloor \frac{l+1}{2} \right\rfloor$ odd power sums. If $n \geq \left\lfloor \frac{l+1}{2} \right\rfloor$ or, equivalently, $2n \geq l$, then these power sums are algebraically independent. Thus an expression for each α_v is uniquely determined, and the above algorithm works. (In fact, the inequality $2n \geq l$ can be strengthened.) For $w_0 \in B_2$, we had $n = 2$ and $l = 4$, so the problem did not come up.

Type B_3 Schubert polynomials $\mathfrak{B}_w^{(3)}$ of the second kind can be obtained by applying the divided difference operators to the top polynomial

$$\begin{aligned}\mathfrak{B}_{w_0}^{(3)}&(x_1,x_2,x_3)\\ &= ((P_1^6-5P_1^3P_3+9P_1P_5-5P_3^2)(-x_1^2x_2+(P_1^3-P_3)/3\\ &\quad -P_1^2x_1+P_1(x_1^2+x_1x_2))/45\\ &\quad -(P_1^8-7P_1^5P_3+14P_1^3P_5+7P_3P_5-15P_1P_7)x_2/105\end{aligned}$$

where $P_k=(x_1^k+x_2^k+x_3^k)/2$ for $k=1,3,5,7$; see (6) for the expansion of this polynomial in the variables x_1, x_2, and x_3.

7.5 Multiplication of the Schubert polynomials. We are now going to show that the Schubert polynomials of the two kinds satisfy condition (2). First note that, in view of the stability property (4) which was proved in Theorems 6.3 and 7.2, it suffices to prove (2) for the power series $\mathfrak{b}_w$ and $\mathfrak{B}_w$ defined by (6.8) and (7.5). In view of Theorem 7.3 (see also Subsection 7.4), the structure constants for the $\mathfrak{b}_w$ and the $\mathfrak{B}_w$ are the same, since they are not affected by the substitution (7.6). Thus it sufices to prove property (2) for the power series $\mathfrak{B}_w$ of the second kind:

$$\mathfrak{B}_u\mathfrak{B}_v=\sum_w c_{uv}^w\mathfrak{B}_w \quad . \tag{7.12}$$

A two-line proof of (7.12) (based on definition (7.8)) was given in [BH]. We reproduce it here to make the paper self-contained.

Recurrences ($1'$) imply that (7.12) holds modulo the ideal I_W of B-symmetric functions. Since both sides of (7.12) belong to the ring $R=\mathbb{C}[x_1,x_2,\dots;p_1,p_3,\dots]$, and the only element in the intersection $R\cap I_W$ is 0, (7.12) follows.

8. Stanley symmetric functions of type B

This section is devoted to studying the basic properties of the type B Stanley symmetric functions H_w defined by (6.2), (4.2), and (4.1).

In the nilCoxeter case, Theorem 4.5 immediately allows us to establish the following connection between the H_w and the ordinary (type A) Stanley symmetric functions G_w.

8.1 Corollary. *Let $\mathbf{x}=(x_1,\dots,x_n)$. Let w be an element of the parabolic subgroup of type A_{n-1} of the hyperoctahedral group B_n that is generated by s_1, ..., s_{n-1}. Then*

$$H_w^{(n)}(\mathbf{x})=G_w^{super}(\mathbf{x},\mathbf{x})$$

where G_w^{super} is the super-symmetric function that canonically corresponds to the stable Schubert polynomial G_w. (In the λ-ring notation, it means that $G_w^{super}(\mathbf{x},\mathbf{y})$ $=G_w(\mathbf{x}+\mathbf{y})$.) □

The last formula implies that, for such w, H_w is a nonnegative integer linear combination of Schur P-functions. (Recall that, by Lemma 6.1, any H_w is an integer linear combination of Schur P-functions.) Tao Kai Lam [TKL1], [TKL2] has recently found a proof that, in fact, H_w is always a *nonnegative* integer combination of P-functions (see also [BH]).

It can be shown that the [skew] P-functions themselves are a special case of the H_w. To do that, we generalize, in a more or less straightforward way, the corresponding S_n-statement about 321-avoiding permutations (see [BJS]).

8.2 Theorem. *Let σ be a skew shifted shape presented in a standard "English" notation (see, e.g., [SS]). Define a "content" of each cell of σ to be the difference between the number of column and the number of row which this cell is in. (For example, the content of a cell lying on the main diagonal is 0.) Read the contents of the cells of σ column by column, from top to bottom; this gives a sequence a_1, ... , a_l. Define an element w_σ of the hyperoctahedral group B_n by $w_\sigma = s_{a_1} \cdots s_{a_l}$ (in fact, a_1, ... , a_l is a reduced decomposition of w_σ). Then $H_{w_\sigma} = P_\sigma$ where P_σ is the skew Schur P-function corresponding to the shape σ.*

One could also ask: which elements $w \in B_n$ can be represented as w_σ (see Theorem 8.2)? The answer (informal though unambigous) is: those w which avoid the following patterns:

$$3\,2\,1 \qquad \overline{3}\,2\,1 \qquad 3\,2\,\overline{1} \qquad \overline{3}\,2\,\overline{1} \qquad 1\,\overline{2} \qquad \overline{1}\,\overline{2}$$

where $\overline{i}$ denotes the element i of a signed permutation that has changed its sign.

Technical proofs of the last statement and of Theorem 8.2 are omitted.

Similarly to the type A case, the symmetric functions $H_w^{(n)}$ can be obtained as some kind of limit of $\mathfrak{b}_w^{(m)}$ as $m \to \infty$.

8.3 Theorem. *For $w \in B_n$,*

$$\lim_{N\to\infty} \mathfrak{b}_w^{(n+N)}(\underbrace{0,\ldots,0}_{N}, x_1,\ldots,x_n) \;=\; H_w^{(n)}(x_1,\ldots,x_n)\ .$$

Furthermore: if $N \geq n-1$, then

$$\mathfrak{b}_w^{(n+N)}(0,\ldots,0,x_1,\ldots,x_n) \;=\; H_w^{(n)}(x_1,\ldots,x_n)\ .$$

Proof. Similar to the proof of Theorem 6.3. □

Here is another useful property of the polynomials H_w .

8.4 Lemma. $H_w^{(n)} = H_{w^{-1}}^{(n)}$.

Proof. Follows from the symmetry of the defining configuration for $H^{(n)}$ (see Figure 7). □

One can also study the Stanley symmetric functions "of the second kind" defined by expanding the symmetric expression $\sqrt{H^{(n)}}$ (cf. Section 7) in the basis of group elements. These symmetric functions can be viewed as stable Schubert polynomials (of the second kind); they clearly are linear combinations of Schur P-functions with rational coefficients.

9. On Schubert polynomials of the third kind

In this section, we introduce a family of polynomials $\mathbf{C}_w^{(n)}$ which we call the type C Schubert polynomials of the third kind. For these polynomials, the respective versions of properties (0) and (1′) are immediate from their definition; we will also prove property (4). Unfortunately, we were unable to prove (3), which would provide a solution to Problem 0-1-3-4 of the Introduction[2]. However, we found significant computational evidence that condition (3) is indeed satisfied by these

[2]This was recently proved by Tao Kai Lam.

polynomials. The type B Schubert polynomials of the third kind $\mathbf{B}_w^{(n)}$ are then defined by

$$\mathbf{B}_w^{(n)} = 2^{-\sigma(w)}\mathbf{C}_w^{(n)} \tag{9.1}$$

where $\sigma(w)$ denotes the number of sign changes in w. It will immediately follow that the $\mathbf{B}_w$ satisfy the type B conditions (0), (1′), and (4), and, under the conjecture stated above, have nonnegative coefficients which are multiples of $2^{-l(w)}$.

First, let us make clear what are the type C divided differences. For $i \geq 1$, they are the same ∂_i as before. For $i = 0$, define

$$\partial_0^C f = \frac{\partial_0 f}{2} = \frac{f(x_1, x_2, \dots) - f(-x_1, x_2, \dots)}{-2x_1} \tag{9.2}$$

which explains (9.1). Now set

$$\mathbf{C}_{w_0}^{(n)} = \mathfrak{b}_{w_0}^{(n)} = \prod_{k=1}^{n} (x_k)^k \prod_{1 \leq i < j \leq n} (x_i + x_j) \tag{9.3}$$

and define the rest of the $\mathbf{C}_w^{(n)}$ by applying the type C divided differences to the top polynomial $\mathbf{C}_{w_0}^{(n)}$ given by (9.3), in accordance with (1′).

9.1 Example. For $n = 2$, we have $\mathbf{C}_{w_0}^{(2)} = x_1x_2^2(x_1 + x_2)$. Applying the divided differences $-\partial_0^C$ and $-\partial_1$ (cf. (1′)), we obtain the following table:

w	$\mathbf{C}_w^{(2)}$
1	1
u_0	$x_1 + x_2$
u_1	x_2
u_0u_1	x_2^2
u_1u_0	$x_1^2 + x_1x_2 + x_2^2$
$u_0u_1u_0$	$x_1x_2(x_1 + x_2)$
$u_1u_0u_1$	x_2^3
w_0	$x_1x_2^2(x_1 + x_2)$

Using (9.1), we then compute the Schubert polynomials of the third kind of type B_2:

w	$\mathbf{B}_w^{(2)}$
1	1
u_0	$(x_1 + x_2)/2$
u_1	x_2
u_0u_1	$x_2^2/2$
u_1u_0	$(x_1^2 + x_1x_2 + x_2^2)/2$
$u_0u_1u_0$	$x_1x_2(x_1 + x_2)/4$
$u_1u_0u_1$	$x_2^3/2$
w_0	$x_1x_2^2(x_1 + x_2)/4$

The table of the type C_3 Schubert polynomials of the third kind is given in Figure 10.

w	$\mathbf{C}_w^{(3)}$
123	1
$\overline{1}23$	$x_1 + x_2 + x_3$
213	$x_2 + x_3$
132	x_3
$2\overline{1}3$	$x_3^2 + x_2^2 + x_2x_3$
$\overline{1}32$	$x_1^2 + x_3x_1 + 2x_3^2 + x_2^2 + x_2x_3$
$\overline{2}13$	$x_1^2 + x_2x_1 + x_3x_1 + x_3^2 + x_2^2 + x_2x_3$
231	x_3^2
312	$x_1^2 + x_3^2 + x_2^2 + x_2x_3$
$\overline{21}3$	$x_1^2x_2 + x_1^2x_3 + x_3^2x_1 + x_1x_2^2 + x_2x_3x_1 + x_3^2x_2 + x_2^2x_3$
$23\overline{1}$	x_3^3
$3\overline{1}2$	$(x_2 + x_3)(x_1^2 + x_3^2 + x_2^2 + x_2x_3)$
$1\overline{2}3$	$x_3^3 + x_3^2x_2 + x_2^3 + x_2^2x_3$
$\overline{2}31$	$x_3(x_1^2 + x_3x_1 + x_3^2 + x_2^2 + x_2x_3)$
321	$x_3(x_1^2 + x_3^2 + x_2^2 + x_2x_3)$
$\overline{3}12$	$x_1^3 + x_1^2x_2 + x_1^2x_3 + x_3^2x_1 + x_1x_2^2 + x_2x_3x_1 + x_3^3 + x_3^2x_2 + x_2^3 + x_2^2x_3$
$\overline{12}3$	$x_3^3x_1 + x_2x_3^3 + x_1^2x_3^2 + 2x_2^2x_3^2 + x_3^2x_2x_1 + x_2^3x_3 + x_2x_1^2x_3 + x_1x_2^2x_3 + x_1^2x_2^2 + x_1x_2^3$
$\overline{2}3\overline{1}$	$x_3^2(x_1^2 + x_3x_1 + x_2^2 + x_2x_3)$
$32\overline{1}$	$x_3^2(x_1^2 + x_3^2 + x_2^2 + x_2x_3)$
$\overline{31}2$	$(x_1 + x_2 + x_3)(x_3^2x_1 + x_3^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1^2x_2 + x_1x_2^2)$
$13\overline{2}$	x_3^4
$3\overline{2}1$	$x_2x_3^3 + x_1^2x_3^2 + 2x_2^2x_3^2 + x_2^3x_3 + x_2x_1^2x_3 + x_1^2x_2^2$
$\overline{3}21$	$x_1^3x_3 + 2x_1^2x_3^2 + x_1^2x_2^2 + x_2x_1^2x_3 + x_3^2x_2x_1 + x_3^3x_1 + x_1x_2^2x_3 + x_3^4 + 2x_2^2x_3^2 + x_2x_3^3 + x_2^3x_3$
$1\overline{3}2$	$x_2x_3^3 + x_2^3x_3 + x_2^4 + x_3^4 + x_2^2x_3^2$
$\overline{1}3\overline{2}$	$x_3^3(x_1^2 + x_3x_1 + x_2^2 + x_2x_3)$
$3\overline{21}$	$x_2x_3(x_2 + x_3)(x_1^2 + x_2x_3)$
$\overline{3}2\overline{1}$	$x_3(x_1^3x_3 + 2x_1^2x_3^2 + x_1^2x_2^2 + x_2x_1^2x_3 + x_3^2x_2x_1 + x_3^3x_1 + x_1x_2^2x_3 + 2x_2^2x_3^2 + x_2x_3^3 + x_2^3x_3)$
$\overline{13}2$	$x_3^4x_1 + x_3^4x_2 + x_3^3x_1^2 + 2x_2^2x_3^3 + x_3^3x_2x_1 + x_2x_1^2x_3^2 + 2x_2^3x_3^2+$ $x_3^2x_1x_2^2 + x_2^4x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2^3 + x_1x_2^4$
$31\overline{2}$	$x_3^3(x_1^2 + x_2x_3 + x_2^2)$
$\overline{32}1$	$x_1x_2x_3(x_1 + x_2 + x_3)^2 + x_1^3x_3^2 + x_1^3x_2^2 + x_3^3x_1^2 + x_1^2x_2^3 + x_2^3x_3^2 + x_2^2x_3^3$
$2\overline{3}1$	$x_3^4x_2 + x_3^3x_1^2 + 2x_2^2x_3^3 + x_2x_1^2x_3^2 + 2x_2^3x_3^2 + x_2^4x_3 + x_1^2x_2^2x_3 + x_1^2x_2^3$
$12\overline{3}$	x_3^5
$3\overline{12}$	$x_2x_3^2(x_2 + x_3)(x_1^2 + x_2x_3)$
$\overline{321}$	$x_1x_2x_3(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)$
$2\overline{31}$	$x_2x_3(x_2 + x_3)^2(x_1^2 + x_2x_3)$
$\overline{1}2\overline{3}$	$(x_1^2 + x_3x_1 + x_2^2 + x_2x_3)x_3^4$
$\overline{3}1\overline{2}$	$(x_1^3x_3 + x_1^2x_3^2 + x_1^2x_2^2 + x_2x_1^2x_3 + x_3^2x_2x_1 + x_1x_2^2x_3 + x_2^2x_3^2 + x_2^3x_3)x_3^2$
$\overline{23}1$	$x_1^2x_3^4 + x_1x_2x_3^4 + x_2^2x_3^4 + x_1^3x_3^3 + 2x_2^3x_3^3 + 2x_1^2x_2x_3^3 + 2x_1x_2^2x_3^3 + 3x_1^2x_2^2x_3^2 + x_2^4x_3^2+$ $x_1^3x_3^2x_2 + 2x_3^2x_1x_2^3 + 2x_1^2x_2^3x_3 + x_1x_2^4x_3 + x_1^3x_2^2x_3 + x_1^2x_2^4 + x_1^3x_2^3$
$21\overline{3}$	$x_3^4(x_1^2 + x_2x_3 + x_2^2)$
$\overline{312}$	$x_1x_2x_3^2(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)$
$\overline{231}$	$x_1x_2x_3(x_2 + x_3)^2(x_1 + x_2)(x_1 + x_3)$
$2\overline{13}$	$x_2x_3^3(x_2 + x_3)(x_1^2 + x_2x_3)$
$1\overline{32}$	$x_2^2x_3^2(x_2 + x_3)(x_1^2 + x_2x_3)$
$\overline{21}\overline{3}$	$x_3^3(x_1^3x_3 + x_1^2x_3^2 + x_1^2x_2^2 + x_2x_1^2x_3 + x_3^2x_2x_1 + x_1x_2^2x_3 + x_2^2x_3^2 + x_2^3x_3)$
$\overline{132}$	$x_1x_2^2x_3^2(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)$
$\overline{213}$	$x_1x_2x_3^3(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)$
$1\overline{23}$	$x_2^2x_3^3(x_2 + x_3)(x_1^2 + x_2x_3)$
$\overline{123}$	$x_1x_2^2x_3^3(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)$

FIGURE 10. Type C_3 Schubert polynomials of the third kind

The polynomials $\mathbf{C}_w^{(n)}$ and $\mathbf{B}_w^{(n)}$ are obviously homogeneous of degree $l(w)$ and, by definition, they satisfy the respective recurrences ($1'$). We are now going to prove the stability property (4); this will also imply that $\mathbf{C}_1^{(n)} = \mathbf{B}_1^{(n)} = 1$, since we have already checked it for $n = 2$.

To prove stability, we need to demonstrate that the rule (9.3) is consistent with the divided difference recurrences. This is shown in the following theorem.

9.2 Theorem. *Let $w = w_0^{(n-1)}$ be the element of maximal length in the parabolic subgroup $B_{n-1} \subset B_n$ generated by $s_1, \dots, s_{n-2}$. Then*

$$\mathbf{C}_w^{(n)}(x_1, \dots, x_{n-1}, 0) = \mathbf{C}_w^{(n-1)}(x_1, \dots, x_{n-1}) \ . \tag{9.4}$$

Proof. Let us observe that the elements of maximal length in B_{n-1} and B_n are related by

$$w_0^{(n)} = w_0^{(n-1)} s_{n-1} s_{n-2} \cdots s_1 s_0 s_1 \cdots s_{n-2} s_{n-1} \ . \tag{9.5}$$

This allows to restate (9.4) as

$$(-\partial_{n-1} \cdots \partial_1 \partial_0^C \partial_1 \cdots \partial_{n-1} f_n)(x_1, \dots, x_{n-1}, 0) = f_{n-1}(x_1, \dots, x_{n-1}) \tag{9.6}$$

where we used the notation

$$f_n(x_1, \dots, x_n) = \prod_{k=1}^{n} (x_k)^k \prod_{1 \le i < j \le n} (x_i + x_j) \tag{9.7}$$

(cf. (9.3)). Keeping in mind that symmetric functions behave as constants with respect to divided differences, we first obtain

$$(-1)^{n-1} \partial_1 \cdots \partial_{n-1} f_n = x_1 x_2 x_3^2 \cdots x_n^{n-1} \prod_{1 \le i < j \le n} (x_i + x_j) = x_1 F$$

where

$$F = x_2 x_3^2 \cdots x_n^{n-1} \prod_{1 \le i < j \le n} (x_i + x_j) \ . \tag{9.8}$$

Then, by the Leibniz rule,

$$(-1)^n \partial_0^C \partial_1 \cdots \partial_{n-1} f_n = F - x_1 \partial_0^C F \ . \tag{9.9}$$

Thus the claim (9.6) can be reformulated as

$$D_{n-1}(F - x_1 \partial_0^C F)\big|_{x_n = 0} = f_{n-1}(x_1, \dots, x_{n-1}) \tag{9.10}$$

where we used the notation

$$D_k = (-1)^k \partial_k \cdots \partial_1 \ .$$

To prove (9.10), we will need the following lemma.

9.3 Lemma.

$$D_{n-1}(x_2 x_3^2 \cdots x_n^{n-1})\big|_{x_n = 0} = x_2 x_3^2 \cdots x_{n-1}^{n-2} \ . \tag{9.11}$$

Proof. Induction on n. For $n=2$, we check that, indeed, $-\partial_1 x_2 = 1$. Suppose we know that $D_{n-2}(x_2x_3^2\cdots x_{n-1}^{n-2}) = x_2x_3^2\cdots x_{n-2}^{n-3} + x_{n-1}f$ for some polynomial f. Then

$$\begin{aligned}
&D_{n-1}(x_2x_3^2\cdots x_n^{n-1})\\
&\quad= -\partial_{n-1}\left(x_n^{n-1}D_{n-2}(x_2x_3^2\cdots x_{n-1}^{n-2})\right)\\
&\quad= -\partial_{n-1}\left(x_2x_3^2\cdots x_{n-2}^{n-3}x_n^{n-1} + x_n^{n-1}x_{n-1}f\right)\\
&\quad= x_2x_3^2\cdots x_{n-2}^{n-3}\left(x_{n-1}^{n-2} + x_{n-1}^{n-3}x_n + \cdots + x_n^{n-2}\right) - x_nx_{n-1}\partial_{n-1}\left(x_n^{n-2}f\right)\ .
\end{aligned}$$

Setting $x_n = 0$, we obtain (9.11), as desired. □

We continue the proof of Theorem 9.2. From (9.8) we get

$$\partial_0^C F = x_2x_3^2\cdots x_n^{n-1}\ \partial_0^C \prod_{1\le i<j\le n}(x_i+x_j)$$

and therefore

$$D_{n-1}(x_1\partial_0^C F) = D_{n-1}(x_1x_2\cdots x_n\, g) = x_1x_2\cdots x_n D_{n-1}\, g$$

for a certain polynomial g, implying

$$D_{n-1}(x_1\partial_0^C F)\big|_{x_n=0} = 0\ . \tag{9.12}$$

Then

$$\begin{aligned}
&D_{n-1}(F - x_1\partial_0^C F)\big|_{x_n=0}\\
&\quad= D_{n-1}F\big|_{x_n=0} && \text{by (9.12)}\\
&\quad= \prod_{1\le i<j\le n}(x_i+x_j)D_{n-1}(x_2x_3^2\cdots x_n^{n-1})\Big|_{x_n=0} && \text{by (9.8)}\\
&\quad= \prod_{1\le i<j\le n}(x_i+x_j)\Big|_{x_n=0}\ \cdot\ x_2x_3^2\cdots x_{n-1}^{n-2} && \text{by (9.11)}\\
&\quad= \prod_{1\le i<j\le n-1}(x_i+x_j)\ \cdot\ x_1x_2^2x_3^3\cdots x_{n-1}^{n-1}\\
&\quad= f_{n-1}(x_1,\dots,x_{n-1})\ , && \text{by (9.7)}
\end{aligned}$$

proving (9.10) and hence Theorem 9.2. □

10. A negative result

10.1 Lemma. *Let $W = B_2$ be the hyperoctahedral group with two generators. Assume that $\{X_w(x_1,x_2)\ :\ w\in W\}$ is a family of polynomials satisfying conditions* (0) *and* (1a) (*see Introduction*) *and the following instances of condition* (2)*: $X_{s_0}^2 = X_{s_1s_0}$, $X_{s_0}X_{s_1} = X_{s_1s_0} + X_{s_0s_1}$. Then*
(a) *for some $w\in W$, the polynomial X_w has both positive and negative coefficients;*
(b) *for some $w\in W$, the polynomial X_w has non-integer coefficients.*

The same statement is true with condition (1) *replaced by* (1′).

Proof. Conditions (0)-(1) imply $X_{s_0} = -\frac{1}{2}(x_1+x_2)$ and $X_{s_1} = -x_2$. We then use (2) to compute $X_{s_0s_1} = X_{s_0}X_{s_1} - X_{s_0}^2 = \frac{1}{4}(x_2^2 - x_1^2)$, which proves both (a) and (b).

The second version of the lemma is equivalent to the first one, under the transformation (7.9). □

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